Abstract: Keisler's theorem and cardinal invariants at uncountable cardinals

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The following is an important theorem on model theory proved by Keisler and Shelah.

Theorem 1 (Keisler–Shelah)**.** For every (first-order) language $\mathcal{L}$ and two $\mathcal{L}$-structures $\mathcal{A}, \mathcal{B}$, the following are equivalent:

(1) $\mathcal{A} \equiv \mathcal{B}$ (that is, $\mathcal{A}$ and $\mathcal{B}$ are elementarily equivalent).

(2) There is a nonprincipal ultrafilter $\mathcal{U}$ over an infinite set such that the ultrapowers $\mathcal{A}^{\mathcal{U}}$ and $\mathcal{B}^{\mathcal{U}}$ are isomorphic.

The following theorem is also known in connection with the above theorem.

Theorem 2 (Keisler, Golshani and Shelah)**.** The following are equivalent:

(1) The continuum hypothesis.

(2) For every countable language $\mathcal{L}$ and two $\mathcal{L}$-structures $\mathcal{A}, \mathcal{B}$ of size $\leq \mathfrak{c}$, if $\mathcal{A} \equiv \mathcal{B}$ then there is a nonprincipal ultrafilter $\mathcal{U}$ over ω such that the ultrapowers $\mathcal{A}^{\mathcal{U}}$ and $\mathcal{B}^{\mathcal{U}}$ are isomorphic.

In order to analyze these theorems in detail, we introduce the following principles.

Definition 3. Let κ, μ and λ be infinite cardinals. We define a principle $\mathrm{KT}^{\mu}_{\kappa}(\lambda)$ by

$$\begin{aligned} \mathrm{KT}^{\mu}_{\kappa}(\lambda) \iff & \text{for every language } \mathcal{L} \text{ of size} \leq \mu \text{ and} \\ & \text{every elementarily equivalent } \mathcal{L}\text{-structures } \mathcal{A}, \mathcal{B} \text{ of size} \leq \lambda, \\ & \text{there is a uniform ultrafilter } \mathcal{U} \text{ on } \kappa \text{ such that } \mathcal{A}^{\mathcal{U}} \simeq \mathcal{B}^{\mathcal{U}}. \end{aligned}$$

We also define a principle $\mathrm{SAT}^{\mu}_{\kappa}(\lambda)$ by

$$\begin{aligned} \mathrm{SAT}^{\mu}_{\kappa}(\lambda) \iff & \text{there is a uniform ultrafilter } \mathcal{U} \text{ on } \kappa \text{ such that} \\ & \text{for every language } \mathcal{L} \text{ of size} \leq \mu \text{ and} \\ & \text{every sequence } \langle \mathcal{A}_i : i < \kappa \rangle \text{ of infinite } \mathcal{L}\text{-structures of size} \leq \lambda, \\ & \text{the ultraproduct } \left(\prod_{i \in \kappa} \mathcal{A}_i \right) / \mathcal{U} \text{ is saturated.} \end{aligned}$$

In this talk, we analyze these principles.

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